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On the Mean-Field Ising Model in a Random External Field

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We use a method developed by van Hemmen to obtain the free energy of the mean-field Ising model in a random external magnetic field. Some results of previous mean-field calculations are confirmed and generalized. The tricritical point in the global phase diagram is discussed in detail. We also consider different probability distributions of the random fields and provide some proofs regarding the conditions for the existence of a tricritical point.

KEY WORDS: Random field; Ising model.

1. INTRODUCTION

In recent times there has been a considerable interest in the study of lattice systems in the presence of random external fields.[1] In particular, the presumed equivalence between ferromagnets in a random field and dilute antiferromagnets in a uniform field has brought some experimental relevance to this problem.[2] The major questions are posed by systems with short-range interactions and discrete internal symmetries, such as the Ising model, for which there has been some very recent progress.[3] However, these systems demand a very hard analysis, and the results obtained so far still preclude a detailed study of the phase diagram. There is thus room for the consideration of less realistic model systems which, however, lend themselves to exact calculations. From this point of view, the spherical model and Curie–Weiss mean-field model, which is the subject of the present paper, will come immediately to mind.

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The spherical model in a random field is amenable to a simple and exact solution.[4] It leads to a lower critical dimensionality $d_l = 4$, which is also conjectured to be correct for systems with continuous internal symmetries. This is in agreement with the fact that the (nonrandom) spherical model is a suitable limit (of "infinite spin dimensionality") of a class of continuous (n-vector) models.[5] The other well-known limiting situation, which may be regarded as a suitable limit of infinite (spatial) dimensionality of models with short-range interactions, is represented by the mean-field theories.[6,7]

In the present paper we discuss a mean-field theory for the ferromagnetic Ising model in a random external field described by the Hamiltonian

$$\mathscr{H} = -\tfrac{1}{2} J N m_1^2 + N m_2 - H N m_1 \tag{1.1}$$

where N is the number of spins, $J > 0$ is the coupling constant, $H > 0$ is the uniform external magnetic field,

$$m_1 \equiv \frac{1}{N} \sum_{i=1}^{N} S_i \tag{1.2}$$

$$m_2 = \frac{1}{N} \sum_{i=1}^{N} h_i S_i \tag{1.3}$$

and $S_i = \pm 1$, for all i. The fields h_i are independent, identically distributed, random variables, with zero mean and nonzero variance. If we denote by E the expectations with respect to the corresponding probability measures, it is assumed that $E(h_i) = 0$, for all i, and $E(h_i h_j) = h^2 \delta_{i,j}$, for all i, j, with $h \neq 0$. As in the case of the spherical model, we anticipate that the existence of ferromagnetic ordering at low temperatures will be controlled by the variance h. It should be remarked, however, that other mean-field models could also be treated along the same lines of this paper.

The model defined in the last paragraph has been considered by Schneider and Pytte,[6] with a Gaussian probability distribution, and by Aharony,[7] with a discrete distribution, from the point of view of "naive mean-field theory." A complete and rigorous solution, however, may be trivially obtained by a simple and elegant method introduced by van Hemmen[8,9] in his treatments of models for a spin glass. This solution, which is briefly discussed in Section 2, involves the consideration of two parameters, m_1 and m_2, instead of just the magnetization m_1. From the conditions for the minimization of the free energy with respect to the parameters, we identify m_1 as the order parameter and m_2 as a nonordering density which may be easily eliminated. In Section 3 we perform a preliminary analysis to regain earlier results for the λ line and the location of a tricritical point in the temperature (T)–variance (h) space. In Section 4 we use the full expression of the free energy to analyze the phase diagram, including the asymptotic behavior of the line of first-order transitions near $T=0$ and the tricritical point. In Section 5 we discuss other probability distributions and provide a simple proof that a tricritical point is absent if the symmetric probability distribution is monotone decreasing along the positive field axis. Also, we show which discrete distributions give rise to a tricritical point and briefly discuss the reason for these qualitative differences. Finally, in Appendix A we discuss an alternative way of formulating the mean-field theory as the limit of "infinite coordination" of a Cayley tree. We proceed on the basis of a paper by Thompson[10] about the Ising spin glass model on a Cayley tree, and show, under some conjectures, that it is possible to regain the results of Section 3. In conclusion, the main interest of the present calculations does not lie on the method, which is a straightforward application of van Hemmen's procedures, but rather on the fact that it is possible to provide analytical proofs of several interesting properties of the phase diagram. This is related to the structure of the equations which are much simpler than their counterparts for the spin glass models.[8,9]

2. THE FREE ENERGY OF THE MODEL

In this section we use the method of van Hemmen[8,9] to obtain the "Gibbs free energy" of the model. Let us define the normalized mean over the 2^N spin configurations,

$$\langle \cdots \rangle_S = \frac{1}{2^N} \sum_{\{S_i\}} \cdots \tag{2.1}$$

and the vector

$$\mathbf{W} = (Nm_1, Nm_2) \tag{2.2}$$

For $\mathbf{t} = (t_1, t_2) \in R^2$, and a fixed configuration, $\{h_i\}$, of the random fields, we calculate the function

$$\begin{aligned} C(\mathbf{t}) &= \lim_{N\to\infty} \frac{1}{N} \ln\langle \exp(\mathbf{t}\cdot\mathbf{W})\rangle_S \\ &= \lim_{N\to\infty} \frac{1}{N} \ln \prod_{i=1}^{N} \cosh(t_1 + t_2 h_i) = E\{\ln[\cosh(t_1 + t_2 h)]\} \end{aligned} \tag{2.3}$$

where the last step holds, with probability one, by the strong law of large numbers. Given the function $C(\mathbf{t})$, we perform the Legendre transformation

$$-C^*(\mathbf{m}) = \sup_{\mathbf{t} \in R^2} [\mathbf{m} \cdot \mathbf{t} - C(\mathbf{t})] \tag{2.4}$$

where $C^*(\mathbf{m})$ plays the role of an entropy (notice that this definition differs from van Hemmen's[9] by a minus sign). The free energy per unit spin in the thermodynamic limit is given by

$$g(\beta) = -\frac{1}{\beta} \max_{\mathbf{m} \in R^2} [Q(\mathbf{m}) + C^*(\mathbf{m})] \tag{2.5}$$

where $\beta = (k_B T)^{-1}$, and

$$Q(\mathbf{m}) = -\frac{\beta \mathscr{H}}{N} = \beta \left(\frac{1}{2} J m_1^2 + H m_1 \right) + \beta m_2 \tag{2.6}$$

In the case of a discrete probability distribution of density

$$p(h_i) = \tfrac{1}{2}[\delta(h_i - h) + \delta(h_i + h)] \tag{2.7}$$

with $h > 0$, we have

$$C(\mathbf{t}) = \tfrac{1}{2} \ln[\cosh(t_1 + t_2 h)] + \tfrac{1}{2} \ln[\cosh(t_1 - t_2 h)] \tag{2.8}$$

It is then easy to write the Gibbs free energy in the form

$$g(T, H, h; m_1, m_2) = -\frac{J}{2} m_1^2 - H m_1 - m_2 - \frac{1}{\beta} C^*(\mathbf{m}) \tag{2.9}$$

where

$$\begin{aligned} C^*(\mathbf{m}) = \ln 2 - \frac{1}{4} \ln \left\{ \left[1 - \left(m_1 + \frac{m_2}{h} \right)^2 \right] \left[1 - \left(m_1 - \frac{m_2}{h} \right)^2 \right] \right\} \\ - \frac{m_1}{4} \ln \frac{(1 + m_1)^2 - (m_2/h)^2}{(1 - m_1)^2 - (m_2/h)^2} \\ - \frac{m_2}{4h} \ln \frac{(1 + m_2/h)^2 - m_1^2}{(1 - m_2/h)^2 - m_1^2} \end{aligned} \tag{2.10}$$

The minimization with respect to the parameters m_1 and m_2 leads to the equation

$$-J m_1 - H - \frac{1}{\beta} \frac{\partial C^*}{\partial m_1} = 0 \tag{2.11}$$

and

$$-1-\frac{1}{\beta}\frac{\partial C^*}{\partial m_2}=0 \tag{2.12}$$

It is convenient to rearrange these equations in the more typical mean-field structure

$$m_1=\tfrac{1}{2}\tanh(\beta H+\beta J m_1+\beta h)+\tfrac{1}{2}\tanh(\beta H+\beta J m_1-\beta h) \tag{2.13}$$

and

$$m_2=\frac{h}{2}\tanh(\beta H+\beta J m_1+\beta h)-\frac{h}{2}\tanh(\beta H+\beta J m_1-\beta h) \tag{2.14}$$

from which we can see that the parameter m_2 is a nonordering density and may be eliminated in terms of the order parameter m_1.

3. PRELIMINARY ANALYSIS OF THE CRITICAL LINE

Since $m_1$is the order parameter of the transition, it is easy to use Eq. (2.13) to obtain an expression for the critical line. This corresponds to the previous mean-field calculations, and this section may be regarded as a recollection of the results of Aharony.[7] In zero uniform field, the right hand side of Eqs. (2.13), (2.14) may be expanded in powers of m_1. Thus we have

$$m_1=\frac{\beta J}{\cosh^2\beta h}m_1-\frac{1}{3}\frac{\cosh^2\beta h-3\sinh^2\beta h}{\cosh^4\beta h}(\beta J)^3 m_1^3+O(m_1^5) \tag{3.1}$$

and

$$m_2=h\tanh\beta h+O(m_1^2) \tag{3.2}$$

In the paramagnetic phase $m_1=0$. In the ferromagnetic phase, near the critical line, T_c, it is possible to write the asymptotic form

$$m_1^2=\frac{1-\beta J\operatorname{sech}^2\beta h}{-\frac{1}{3}(-2\operatorname{sech}^2\beta h+3\operatorname{sech}^4\beta h)(\beta J)^3} \tag{3.3}$$

if we assume that $m_1\to 0$ as $T\to T_c^-$. Then we have the following conditions for the existence of a ferromagnetic phase:

$$1-\beta J\operatorname{sech}^2\beta h<0 \tag{3.4}$$

and

$$-\tfrac{1}{3}(-2\,\mathrm{sech}^2\,\beta h + 3\,\mathrm{sech}^4\,\beta h)(\beta J)^3 < 0 \tag{3.5}$$

The critical line is given by the expression

$$\beta J = \cosh^2 \beta h \tag{3.6}$$

supplemented by the stability condition (3.5). If we insert (3.6) into (3.5), we obtain $2\beta J < 3$, which leads to a tricritical point at $\beta J = \frac{3}{2}$, since the violation of (3.5) signals the onset of a first-order transition.

In Fig. 1 we show the phase diagram in terms of the reduced field variables $g = h/J$ and $t = (\beta J)^{-1}$. The λ line, given by Eq. (3.6), is denoted by the dashed curve. The line of first-order transitions, which can be

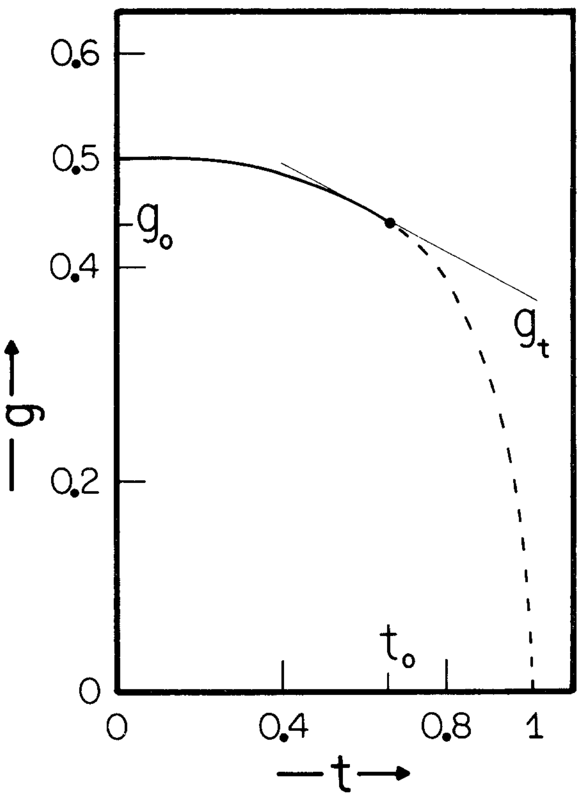

Fig. 1. Phase diagram, in the $g = h/J$, $t = (\beta J)^{-1}$ space, showing a tricritical point [$t_0 = \frac{2}{3}$ and $g_0 = \frac{2}{3}\cosh^{-1}(\frac{3}{2})^{1/2}$] at the junction of a first-order boundary (solid curve) and a line of critical points (dashed curve). The common tangent at the tricritical point is also indicated.

numerically calculated from the free energy, is denoted by the solid line. As we shall see in Section 5, the existence of the tricritical point is related to the nature of the probability distribution. The critical value $g=0.5$ at $T=0$, together with the asymptotic exponential behavior of the boundary near $T=0$, will be obtained in the next section. It should be remarked that the same expression for the λ line, and the location of the tricritical point, can be obtained through the use of bifurcation theory[9] to analyze Eqs. (2.13) and (2.14).

4. THE GLOBAL PHASE DIAGRAM

All features of the phase diagram are already contained in Eq. (2.9), supplemented by the conditions given by Eqs. (2.11) and (2.12). In this section, let us keep the order parameter $m_1$fixed and use Eq. (2.12) to eliminate the nonordering parameter m_2. From Eq. (2.12) we have

$$\frac{1}{4}\ln\frac{1+m_1+m_2/h}{1-m_1-m_2/h}-\frac{1}{4}\ln\frac{1+m_1-m_2/h}{1-m_1+m_2/h}=\beta h \tag{4.1}$$

Then it is easy to write

$$\frac{m_2}{h}=\frac{1}{\tanh 2\beta h}\{1\pm[1-(1-m_1^2)\tanh^2 2\beta h]^{1/2}\} \tag{4.2}$$

At $m_1=0$, the plus root yields a divergence for $\beta h\to 0$. We should then use the minus root to obtain a physical expression for m_2 in terms of m_1.

Let us expand the right-hand side of Eq. (4.2) in powers of m_1:

$$\begin{aligned}\frac{m_2}{h}={}&\tanh\beta h-\frac{1}{2}\sinh 2\beta h\, m_1^2+\frac{1}{8}\sinh^3 2\beta h\, m_1^4\\&-\frac{1}{16}\sinh^5 2\beta h\, m_1^6+\cdots\end{aligned} \tag{4.3}$$

Now it is straightforward to use this expansion to eliminate m_2, and to obtain an expansion of the free energy, given by Eq. (2.9), as a power series in the order parameter m_1. In zero uniform field we have

$$g(T, H=0, h; m_1)=A+Bm_1^2+Cm_1^4+Dm_1^6+\cdots \tag{4.4}$$

where A, B, C, D are smooth functions of T and h, given in Appendix B. It should be remarked that Eq. (4.4) corresponds to the usual Landau picture of phase transitions. The critical line is determined by the conditions $B=0$, and $C>0$, which are equivalent to Eq. (3.6) supplemented by the stability requirement given by Eq. (3.5). The tricritical point is located by $B=C=0$, with $D>0$, which also corresponds to the conjecture of the last section. Needless to say, these results are supported by an analysis of Eqs. (2.13) and (2.14) using bifurcation theory.

The calculations of the last paragraph do not yield an analytic expression for the line of first-order transitions. However, we can easily obtain asymptotic results near $T=0$ and near the tricritical point. At $T=0$, Eq. (2.9) may be written as

$$g(T=0, H=0, h; m_1, m_2) = -\frac{J}{2} m_1^2 - m_2 \tag{4.5}$$

where we should keep in mind that $m_1 \in [-1, 1]$, and $m_2 \in [-h, h]$. The limiting behavior of Eqs. (2.13) and (2.14) depends on $g=h/J$. For $g \geqslant 1$, $m_1 \to 0$, and $m_2 \to h$, which corresponds to the paramagnetic phase. For $g<1$, there are three possibilities: (i) if $m_1 > g$, then $m_1 \to 1$, and $m_2 \to 0$; (ii) if $m_1 < -g$, then $m_1 \to -1$, and $m_2 \to 0$; (iii) if $-g \leqslant m_1 \leqslant g$, then $m_1 \to 0$, and $m_2 \to h$. To decide among these possibilities, we have to look at the free energy. Cases (i) and (ii) yield $g(T=0,\ H=0,\ h; m_1 = \pm 1, m_2 = 0) = -J/2$. Case (iii) yields $g(T=0,\ H=0,\ h; m_1 = 0,\ m_2 = h) = -h$. Then, if $h < J/2$, or $g < 1/2$, we have the ferromagnetic phase. Near $g=1/2$, it is easy to show that the phase boundary is given by the usual law

$$g = \frac{1}{2} - 0\left[t \exp\left(-\frac{3}{2t}\right)\right] \tag{4.6}$$

The asymptotic behavior of the first-order boundary near the tricritical point, $t_0 = 2/3$, $y_0 = (2/3)\cosh^{-1}(3/2)^{1/2}$, can be obtained from the expansion (4.4). Indeed, it is easy to see that the relation

$$C^2 = 4BD \tag{4.7}$$

represents a necessary asymptotic condition for the coexistence of the ordered and disordered phases. As usual,[11] let us write the following expansion about the tricritical point:

$$B = B_1 \Delta t + B_2 \Delta g \tag{4.8}$$

$$C = C_1 \Delta t + C_2 \Delta g \tag{4.9}$$

and

$$D = D_0 > 0 \tag{4.10}$$

where $\Delta t = t - t_0$, $\Delta g = g - g_0$, and the expressions of the coefficients are given in Appendix B. At the tricritical point, the common tangent is given by

$$g_t = g_0 - \frac{B_1}{B_2} \Delta t \tag{4.11}$$

Then it is more convenient to introduce the variable $\Delta\tilde{g} = g - g_t$, in terms of which we have

$$B = B_2 \Delta\tilde{g} \tag{4.12}$$

and

$$C = \left(C_1 - \frac{B_1 C_2}{B_2} \right) \Delta t + C_2 \Delta\tilde{g} \tag{4.13}$$

Inserting these expressions into Eq. (4.7), we have the asymptotic form

$$\Delta\tilde{g} = \frac{(B_2 C_1 - B_1 C_2)^2}{4B_2^3} \Delta t^2 \tag{4.14}$$

Since, from Appendix B, $B_2 C_1 - B_1 C_2 \neq 0$ and $B_2 > 0$, this result yields an asymptotic first-order phase boundary which rises above the common tangent near the tricritical point.

Finally, it should be remarked that there is no difficulty to use the same procedures to analyze the complete phase diagram, in terms of the variables H, T, and h. As usual, for $H \neq 0$, there will be two wing-shaped first-order surfaces ending at λ lines which join smoothly at the tricritical point.

5. ON THE EXISTENCE OF THE TRICRITICAL POINT

The existence of a tricritical point depends on the nature of the probability distribution of the random fields. If the results of Appendix B are generalized to include other probability distributions, it is possible to see that we may obtain the stability criteria along the lines described in Section 3. That is, we first expand the analog of Eq. (2.13) in powers of the order parameter,

$$m_1 = E(\operatorname{sech}^2 \beta h)\, \beta J m_1 - \frac{2}{3} E\left[\frac{-2\cosh^2 \beta h + 3}{\cosh^4 \beta h} \right] (\beta J)^3 m_1^3 + \cdots \tag{5.1}$$

Then, there will be no tricritical point (that is, the λ line will be always stable) if

$$E(3\,\mathrm{sech}^4\,\beta h - 2\,\mathrm{sech}^2\,\beta h) \geqslant 0 \tag{5.2}$$

which is the anolog of the stability conditions of the previous sections. We now formulate some conditions on the probability distributions.

Assumption 1. The probability distribution is absolutely continuous and even, and its density p is monotone decreasing in $[0, \infty]$.

Proposition 5.1. Under Assumption 1 the inequality (5.2) holds.

Proof. Let us write

$$y(x) \equiv 3\,\mathrm{sech}^4\,x - 2\,\mathrm{sech}^2\,x \tag{5.3}$$

Since $y(x)$ is even we have

$$\int_{-\infty}^{+\infty} y(x)\,dx = 2\int_0^{\infty} y(x)\,dx \tag{5.4}$$

Now, notice that

$$\begin{aligned}\int_0^{\infty} dx\,\mathrm{sech}^4\,x &= \int_0^{\infty} dx\,\mathrm{sech}^2\,x\,\frac{d}{dx}(\tanh x)\\ &= 2\int_0^{\infty} dx\,\mathrm{sech}^2\,x\,\tanh^2 x = 2\int_0^{\infty} dx\,\tanh^2 x\,\frac{d}{dx}(\tanh x)\\ &= 2 - 4\int_0^{\infty} dx\,\mathrm{sech}^2\,x\,\tanh^2 x \end{aligned} \tag{5.5}$$

Hence

$$\int_0^{\infty} dx\,\mathrm{sech}^2\,x\,\tanh^2 x = \tfrac{1}{3} \tag{5.6}$$

and

$$\int_0^{\infty} y(x)\,dx = 0 \tag{5.7}$$

Owing to this remarkable property, and to the form of the function $y(x)$,

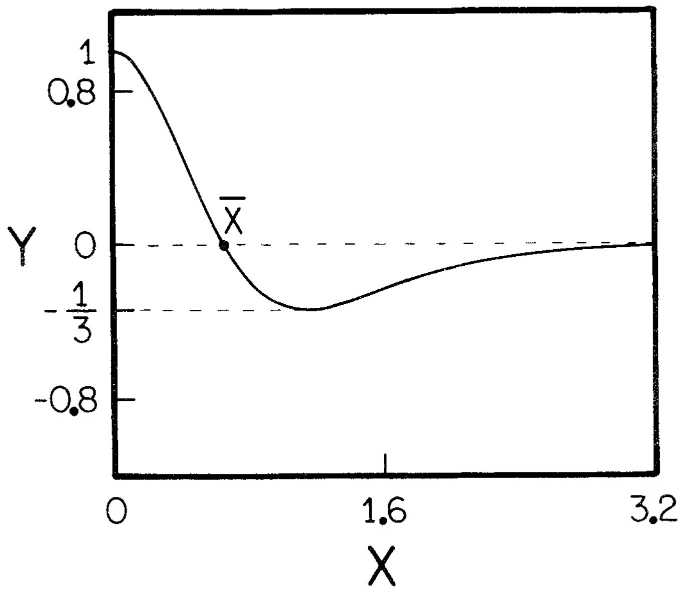

Fig. 2. Sketch of the function $y(x)$.

which is depicted in Fig. 2, we can give a simple proof of the proposition. Let us write

$$I = E(3\,\mathrm{sech}^4\,\beta h - 2\,\mathrm{sech}^2\,\beta h) = \int_{-\infty}^{+\infty} dx\ p(x)\ y(\beta x) \tag{5.8}$$

Since p and y are even functions, and

$$\alpha = \int_0^{\bar{x}} dx\ y(x) = -\int_{\bar{x}}^{\infty} dx\ y(x) \tag{5.9}$$

we have

$$\begin{aligned} I &= \frac{1}{\beta}\, p\left(\frac{x_1}{\beta}\right) \int_0^{\bar{x}} dx\ y(x) + \frac{2}{\beta}\, p\left(\frac{x_2}{\beta}\right) \int_{\bar{x}}^{\infty} dx\ y(x) \\ &= \frac{2}{\beta}\left[p\left(\frac{x_1}{\beta}\right) - p\left(\frac{x_2}{\beta}\right)\right] \int_0^{\bar{x}} dx\ y(x) \end{aligned} \tag{5.10}$$

where $x_1 \in (0, \bar{x})$, and $x_2 \in (\bar{x}, \infty)$. As p is monotone decreasing we have $I \geqslant 0$. ∎

It should be remarked that Assumption 1 includes the case of a Gaussian distribution as treated by Schneider and Pytte.[6]

Assumption 2. Let us assume

$$p(x) = C_0\delta(x) + \frac{1}{2}\sum_i C_i[\delta(x+x_i) + \delta(x-x_i)] \tag{5.11}$$

where $C_i \geqslant 0$, for all i, $C_0 + \sum_i C_i = 1$, and $0 < x_1 < x_2 < \cdots$.

Proposition 5.2. Under Assumption 2, the inequality (5.2) holds for $C_0 - \frac{1}{3}\sum_i C_i \geqslant 0$.

Proof. With the probability distribution (5.11) we can write

$$I = C_0 + \sum_i C_i\, y(\beta x_i) \tag{5.12}$$

Since $y(x) \geqslant -1/3$, we have

$$I \geqslant C_0 - \frac{1}{3}\sum_i C_i \tag{5.13}$$

Thus, $I \geqslant 0$ for $C_0 - \frac{1}{3}\sum_i C_i \geqslant 0$, and the proposition is proved. ∎

There are some physical reasons to explain the qualitative differences between the probability distributions satisfying either Assumption 1 or Assumption 2. This is easier to see if we write Hamiltonian (1.1) in the equivalent form

$$\mathscr{H} = -\frac{J}{N}\sum_{i,j}\left(\frac{h_i h_j}{h^2}\right) S_i' S_j' - h\sum_i S_i' \tag{5.14}$$

where $S_i' = \pm 1$ for all i. The mean-field interactions are long ranged, but a discrete distribution of probabilities samples just a few values of the couplings and, thus, introduces some short-ranged elements into the problem.[9] From this point of view, we might say that discrete distributions have a closer connection with real materials. On the other hand, Gaussian distributions sample many values of the couplings, and work to reinforce the long range nature of the interactions.

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APPENDIX A: ISING MODEL IN A RANDOM FIELD ON A CAYLEY TREE

In this Appendix we show the relation, under some conjectures, between the mean-field model and a suitable limit (of "infinite coordination number") of an Ising model of spins on a Cayley tree. The notation and the details of this calculation are based on a work by Thompson[10] on the Ising spin glass model. Let $\langle \sigma_0 \rangle$ and $\langle \sigma_j \rangle$ denote the thermal expectation values of a spin on the central site and on a site of the first shell of a Cayley tree, respectively. Then we have

$$\begin{aligned}\langle \sigma_j \rangle = {} & \tfrac{1}{2}[\tanh(K+L_j) - \tanh(K-L_j)] \\ & + \tfrac{1}{2}[\tanh(K+L_j) + \tanh(K-L_j)]\langle \sigma_0 \rangle \end{aligned} \tag{A1}$$

where J is the coupling constant, $K=\beta J$, and the L_j's satisfy the set of coupled equations

$$\langle \sigma_0 \rangle = \tanh\left[B + C_0 + \sum_{i_1=1}^{z} \tanh^{-1}(X_{i_1} \tanh K)\right] \tag{A2a}$$

$$X_{i_1} = \tanh L_{i_1} = \tanh\left[B + C_{i_1} + \sum_{i_2=1}^{z-1} \tanh^{-1}(X_{i_1,i_2} \tanh K)\right] \tag{A2b}$$

$$\begin{aligned} X_{i_1,i_2} = \tanh L_{i_1,i_2} = \tanh\Bigg[& B + C_{i_1,i_2} \\ & + \sum_{i_3=1}^{z-1} \tanh^{-1}(X_{i_1,i_2,i_3} \tanh K)\Bigg] \end{aligned} \tag{A2c}$$

$$\vdots$$

$$X_{i_1,i_2,\ldots,i_N} = \tanh[B + C_{i_1,i_2,\ldots,i_N}] \tag{A2d}$$

for a Cayley tree with N shells and coordination z, in a uniform field H, with $B=\beta H$. In these equations $C_{i_1,i_2,\ldots,i_j} \equiv \beta h_{i_1,i_2,\ldots,i_j}$, where $h_{i_1,i_2,\ldots,i_j}$ is the random field in the spin at the site $(i_1, i_2,\ldots, i_j)$ of the jth shell. Let us write $K=K_0/z$, and take the limit $z \to \infty$. We thus have

$$\langle \sigma_0 \rangle_z = \tanh\left(B + C_0 + \frac{K_0}{z} \sum_{i=1}^{z} X_i\right) \tag{A3}$$

and

$$\langle \sigma_j \rangle = \tanh L_j = X_j \tag{A4}$$

Now it is crucial to assume that

$$\lim_{z \to \infty} \frac{1}{Z} \sum_{j=1}^{z} \langle \sigma_j \rangle = E(\langle \sigma_1 \rangle) \tag{A5}$$

and

$$\lim_{z \to \infty} \langle \sigma_0 \rangle_z = E(\langle \sigma_0 \rangle) \tag{A6}$$

where E denotes the expectation with respect to the probability measure describing the random fields. These assumptions of "self-averaging," which should be true because of the reproducibility of the measurements,(12) are indeed satisfied by the magnetization per spin in some random systems as a consequence of the ergodic theorem. However, owing to the residual surface effects in the Cayley tree, we were unable to prove them here. Using Eqs. (A5) and (A6) we have

$$E(\langle \sigma_0 \rangle) = E\{\tanh[B + C_0 + K_0 E(\langle \sigma_1 \rangle)]\} \tag{A7}$$

which is equivalent to Eq. (2.13) if we make the additional assumption of the Bethe approximation, that is, $E(\langle \sigma_0 \rangle) = E(\langle \sigma_1 \rangle)$. Equation (A7) could also be regarded as a mapping, whose fixed point corresponds to the magnetization per spin "deep inside the lattice."

APPENDIX B: COEFFICIENTS OF THE FREE ENERGY

The coefficients of the expansion of the Gibbs free energy in terms of the order parameter, Eq. (4.4), are given by

$$A = -\frac{1}{\beta} \ln 2 - \frac{1}{\beta} \ln(\cosh \beta h) \tag{B1}$$

$$B = -\frac{1}{2} J + \frac{1}{2\beta} \cosh^2(\beta h) \tag{B2}$$

$$C = \frac{1}{12\beta} \cosh^6 \beta h[1 - 3 \tanh^2 \beta h] \tag{B3}$$

$$D = \frac{1}{6\beta} \cosh^6 \beta h \left[\frac{6}{5} \cosh^4 \beta h - 3 \cosh^2 \beta h + 2\right] \tag{B4}$$

The coefficients of the asymptotic forms in the neighborhood of the tricritical point, Eqs. (4.8)–(4.10), are given by

$$B_1 = \frac{3}{4} J - \frac{3\sqrt{3}}{4} h_0, \qquad B_2 = \frac{\sqrt{3}}{2} J \tag{B5}$$

$$C_1 = \frac{9\sqrt{3}}{16} h_0, \qquad C_2 = -\frac{3\sqrt{3}}{8} J, \qquad D_0 = \frac{3}{40} J \tag{B6}$$

From these expressions, we can see that $D_0 > 0$, $B_2 C_1 - B_1 C_2 \neq 0$, and $B_2 > 0$.

REFERENCES

1. Y. Imry and S. K. Ma, *Phys. Rev. Lett.* **35**:1399 (1975); see also E. Pytte, Y. Imry, and D. Mukamel, *Phys. Rev. Lett.* **46**:1173 (1981) and G. Grinstein and S. K. Ma, *Phys. Rev. Lett.* **49**:685 (1982).
2. S. Fishman and A. Aharony, *J. Phys. C* **12**:L729 (1979); see the experimental works by H. Yoshizawa, R. A. Cowley, G. Shirane, R. J. Birgeneau, H. J. Guggenheim, and H. Ikeda, *Phys. Rev. Lett.* **48**:438 (1982), and by D. P. Belanger, A. R. King, and V. Jaccarino, *Phys. Rev. Lett.* **48**:1050 (1982).
3. J. Z. Imbrie, *Phys. Rev. Lett.* **53**:1747 (1984).
4. L. A. Pasthur, *J. Stat. Phys.* **27**:119 (1982); J. F. Perez, W. F. Wreszinski, and J. L. van Hemmen, *J. Stat. Phys.* **35**:89 (1984).
5. H. E. Stanley, *Phys. Rev.* **176**:718 (1968).
6. T. Schneider and E. Pytte, *Phys. Rev. B* **15**:1519 (1977).
7. A. Aharony, *Phys. Rev. B* **18**:3318 (1978).
8. J. L. van Hemmen, *Phys. Rev. Lett.* **49**:409 (1982).
9. J. L. van Hemmen, A. C. D. van Enter, and J. Canisius, *Z. Phys. B* **50**:311 (1983).
10. C. J. Thompson, *J. Stat. Phys.* **27**:457 (1982).
11. R. B. Griffiths, *Phys. Rev. B* **7**:545 (1973).
12. A. C. D. van Enter and J. L. van Hemmen, *Phys. Rev. A* **29**:355 (1984).